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# Dynamical systems related to the Cremmer-Gervais $R$ -matrix

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## Abstract

The generalized Cremmer-Gervais  $R$ -matrix being a twist of the standard  $R$ -matrix of  $SL_q(3)$ , depends on two extra parameters. Properties of this  $R$ -matrix are discussed and two dynamical systems, the quantum group covariant  $q$ -oscillator and an integrable spin chain with a non-hermitian Hamiltonian, are constructed.

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# 1 Introduction

A variety of integrable models of the quantum field theory, classical statistical mechanics and quantum mechanics (systems with finite degrees of freedom) are studied using the quantum inverse scattering method (QISM) (see e.g. [1]-[3]). The main ingredient or a cornerstone of their solution in the framework of the QISM is the  $R$ -matrix, a solution to the Yang-Baxter equation (YBE). Although there is plenty of known solutions, in particular, corresponding to any simple Lie (super)algebra [2], [4], [5], any new  $R$ -matrix, even being connected one way or another to the known ones gives rise to a new set of integrable models. A simple method to get "new"  $R$ -matrices from the standard ones (constant or spectral-parameter-dependent) is the twist procedure [6], [7]. The multiparametric  $R$ -matrix corresponding to the algebra  $gl(n)$  with  $\frac{1}{2}n(n-1)$  extra parameters on the diagonal, was obtained from the standard  $sl_q(n)$   $R$ -matrix (  $\omega = q - q^{-1}$  )

$$R(q) = \sum_{i=1}^n q e_{ii} \otimes e_{ii} + \sum_{i=j} e_{ii} \otimes e_{jj} + \sum_{i<j} \omega e_{ij} \otimes e_{ji} , \quad (1.1)$$

where  $(e_{ij})_{ab} = \delta_{ia}\delta_{jb}$  are basis matrices, by the twist  $F$  constructed from the generators of the Cartan subalgebra [7]

$$R(q, \{p_{ij}\}) = F_{21} R(q) F_{12}^{-1}, \quad F \in \mathcal{U}(h) \otimes \mathcal{U}(h).$$

In this paper we shall look for integrable systems related to a particular  $R$ -matrix, so we shall write formulas in some fixed irreducible representations and not in the universal algebraic form of the quantum group theory [5], [8].

Recently it was shown that the  $R$ -matrix  $R_{CG}(q)$  derived for the quantum Toda theory [9] is a twist of the standard  $R$ -matrix of the  $sl_q(n)$  [10], [11]. We shall restrict ourselves to the case of the  $sl(3)$  algebra.

An interesting feature of this generalized Cremmer-Gervais  $R$ -matrix  $R(q, p, \nu)$  (the twist adds two more parameters to the original solution [9] of the YBE) consists of two new non-zero entries with respect to the standard  $R(q)$

$$\begin{aligned} R(q, p, \nu) = & R(q) + (p-1)(e_{11} \otimes e_{22} + e_{22} \otimes e_{33}) \\ & + (p^{-1}-1)(e_{22} \otimes e_{11} + e_{33} \otimes e_{22}) + (p^2/q-1)e_{11} \otimes e_{33} \\ & + (q/p^2-1)e_{33} \otimes e_{11} + q\nu(e_{32} \otimes e_{12} - p^2/q^2 e_{12} \otimes e_{32}). \end{aligned} \quad (1.2)$$

Due to these extra entries proportional to  $\nu$  the quantum space  $\mathbf{C}_q^3$  commutation relations for (1.2), which were obtained already in [9], are more complicated than for the  $sl(3)$  case:  $x_i x_j = q_{ij} x_j x_i$ ,  $1 \leq i < j \leq 3$ . They coincide with a deformed Heisenberg algebra (  $q$ -oscillator )  $\mathcal{A}_q$

$$x_2 x_1 = p q x_1 x_2, \quad x_2 x_3 = (p q)^{-1} x_3 x_2, \quad x_1 x_3 - p^{-2} x_3 x_1 = \nu x_2^2 \quad (1.3)$$

after the identification  $x_1 = A, x_2 = K, x_3 = A^\dagger$  with some set of  $\mathcal{A}_q$  generators (Sec.3). Looking at a quantum group covariance property of this system:  $x_i \rightarrow \sum_j T_{ij} \otimes x_j$  and studying the corresponding differential calculus [12] the  $R$ -matrix (1.2) was found also in [13].

There is no quadratic element (Hamiltonian) in  $\mathcal{A}_q$  invariant with respect to the quantum group  $\mathcal{A}(R)$  coaction:  $\varphi(x_i) = \sum_j T_{ij} \otimes x_j$ . Hence, after coaction one gets more complicated Hamiltonian in a tensor product of representation spaces of  $\mathcal{A}(R)$  and  $\mathcal{A}_q$ . We analyze in this paper the physical consequences of this transformation for a particular Hamiltonian according to the approach discussed in [14], [15].

The properties of the generalized Cremmer-Gervais  $R$ -matrix  $R(q, p, \nu)$  and the corresponding quantum group  $\mathcal{A}(R)$  and quantum algebra (a dual to the Hopf algebra  $\mathcal{A}(R)$ ) are discussed in Sec.2. The quantum group  $\mathcal{A}(R)$  and quantum algebra corresponding to the original  $R_{CG} = R(q, q^{1/3}, \nu)$  were studied already in [16]. The next section deals with the  $q$ -oscillator interpretation of the quantum space  $\mathbf{C}_q^3(R(q, p, \nu))$ . The different values of the deformation ( $q$ ) and twist ( $p, \nu$ ) parameters correspond to the different choice of the  $q$ -oscillator generators [17, 18]. An integrable spin-chain Hamiltonian is discussed in Sec.4. Although this Hamiltonian is not Hermitian due to the inequality  $(\mathcal{P}R_{CG})^\dagger \neq \mathcal{P}R_{CG}$ , its structure coincides with the one of [19] where non-Hermitian Hamiltonians were used to describe some kinetic processes. Let us mention also a couple of recent papers [20, 22, 23] demonstrating an active interest in different applications of the Drinfeld twist.

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## 2 Properties of the generalized Cremmer-Gervais $R$ -matrix

Although in the framework of the quantum group theory one uses such general objects as universal  $R$ -matrix  $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$ , where  $\mathcal{A}$  is a quasitriangular Hopf algebra [5] and universal twist element  $\mathcal{F} \in \mathcal{A} \otimes \mathcal{A}$ , with appropriate properties (see, e.g. [6, 7], we will be working with finite dimensional matrices which are images of  $\mathcal{R}$  and  $\mathcal{F}$  in corresponding representations  $\rho, \pi$  of the quasi-triangular Hopf algebra  $\mathcal{A}$

$$R = (\rho \otimes \pi)\mathcal{R}, \quad F = (\rho \otimes \pi)\mathcal{F}. \quad (2.1)$$

A twisted  $R$ -matrix  $\mathcal{R}^{(\mathcal{F})}$  is given by the twist transformation [6, 7]

$$\mathcal{R}^{(\mathcal{F})} = \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1} \quad (2.2)$$

of the original  $R$ -matrix  $\mathcal{R}$ , where  $\mathcal{F}_{21} = \mathcal{P}\mathcal{F}\mathcal{P}$ , and  $\mathcal{P}$  is the permutation map in  $\mathcal{A} \otimes \mathcal{A}$ . The twist element satisfies the relations in  $\mathcal{A} \otimes \mathcal{A}$  [6, 7]

$$(\epsilon \otimes id) \mathcal{F} = (id \otimes \epsilon) \mathcal{F} = 1,$$

and in  $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$

$$\mathcal{F}_{12} (\Delta \otimes id) \mathcal{F} = \mathcal{F}_{23} (id \otimes \Delta) \mathcal{F}.$$

In the fundamental representation of the  $sl(3)$  algebra the generalized Cremmer-Gervais  $R$ -matrix  $R_{CG}$  is defined by the standard  $R(q)$  (1.1) and the twist matrix  $F$  [11]

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q/p & 0 & p\nu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & p & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad F_{21} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p\nu & 0 & q/p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (2.3)$$

depending on two arbitrary parameters  $p$  and  $\nu$

$$R(q, p, \nu) = F_{21} R(q) F^{-1}. \quad (2.4)$$

A twist transformation of  $R$ -matrix related to the braid group  $\check{R} = \mathcal{P}R$ ,

$$\check{R}^{(F)} = \mathcal{P}R^{(F)} = \mathcal{P}F_{21}RF^{-1} = F\check{R}F^{-1}, \quad (2.5)$$

is a similarity transformation. Hence, the spectral characteristics of  $\check{R}$  and  $\check{R}^{(F)}$  are the same. In particular  $\check{R}_{CG}$  also satisfies the Hecke condition ( $\omega = q - q^{-1}$ )

$$\begin{aligned} \check{R}^2 &= I + \omega \check{R}, \\ \check{R} &= qP^{(+)} - q^{-1}P^{(-)}, \end{aligned} \quad (2.6)$$

where  $P^{(\pm)}$  are projectors of  $rank P^{(+)} = n(n+1)/2$  and  $rank P^{(-)} = n(n-1)/2$ ,  $n = 3$  for the case of  $sl(3)$ , we are interested in.

Using the FRT approach [8], the standard notations of the QISM and a particular  $R$ -matrix, one can define a quantum group  $\mathcal{A}(R)$  with a matrix  $T = \{T_{ij}\}$  of the generators satisfying the FRT-relation

$$R_{12}T_1T_2 = T_2T_1R_{12}, \quad \text{or} \quad [\check{R}, T \otimes T] = 0. \quad (2.7)$$

The quantum group  $\mathcal{A}(R)$  associated with the original Cremmer-Gervais  $R$ -matrix was defined and discussed already in [9] and [16]. In particular, the constructed quantum determinant was central. In the case of  $R(q, p, \nu)$  (2.4) the quantum determinant  $\det_q T$  is not central. It can be defined by fusion procedure [2], projecting  $T^{\otimes 3} = T_1 T_2 T_3$  to one-dimensional space of the  $q$ -antisymmetrizer

$$P_{123}^{(-)} \simeq P_{12}^{(-)} \left( \left( q + \frac{1}{q} \right) P_{23}^{(-)} - I \right) P_{12}^{(-)}. \quad (2.8)$$

The Yang - Baxter equation for  $R_{12}$  and the defining relations (2.7) mean that the blocks  $R_{ik,jl}$  with fixed  $i, j$  give rise to a representation of the  $\mathcal{A}(R)$  generators  $T_{ij}$ . The corresponding quantum determinant of  $R(q, p, \nu)$  is not proportional to the identity  $3 \times 3$  matrix although it is diagonal

$$\det_q R(q, p, \nu) = q \operatorname{diag} \left( \frac{q}{p^3}, 1, \frac{p^3}{q} \right). \quad (2.9)$$

Projecting the commutation relation of  $T_4 = I^{\otimes 3} \otimes T$  with  $T^{\otimes 3} = T_1 T_2 T_3 = T \otimes T \otimes T \otimes I$

$$\prod_{j=1}^3 R_{j4}(T^{\otimes 3} T_4) = (T_4 T^{\otimes 3}) \prod_{j=1}^3 R_{j4}, \quad (2.10)$$

and using the  $q$ -antisymmetrizer  $P_{123}^{(-)}$  (2.8), one gets a compact matrix form of the commutation relations between  $\det_q T$  and  $T_{ij}$ , provided the former is invertible

$$(\det_q T) T (\det_q T)^{-1} = (\det_q R)^{-1} T (\det_q R), \quad (2.11)$$

where the r.h.s. is the usual matrix product of  $3 \times 3$  matrices  $T$  and (2.9).

For discussing a quantum group  $\mathcal{A}(R)$  coaction on quantum spaces, it is important to have a  $*$ -operation on  $\mathcal{A}(R)$  and on the quantum space. Taking the deformation parameters real,  $q, p, \nu \in \mathbf{R}$  and applying a  $*$ -operation to the defining FRT-relation (2.7) one gets

$$R_{12} T_2^* T_1^* = T_1^* T_2^* R_{12}, \quad (2.12)$$

where  $T^* = \{(T_{ij})^*\}$ . By the similarity transformation of this relation with matrices  $\mathcal{P}$  and  $C \otimes C$ , where  $C_{ij} = \delta_{i,4-j}$ , the relation (2.12) is reduced to the original form (2.7)

$$R_{12} \tilde{T}_1^* \tilde{T}_2^* = \tilde{T}_2^* \tilde{T}_1^* R_{12}, \quad \tilde{T}^* = C T^* C. \quad (2.13)$$

Hence, the  $*$ -operation on  $\mathcal{A}(R_{CG})$  can be defined as follows (cf. [13])

$$T^* = C T C. \quad (2.14)$$

This form of the  $*$ -operation is valid also for the quantum group in the  $SL(n)$  case with the matrix  $C_{ij} = \delta_{i,n+1-j}$ . The corresponding quantum determinant of the  $R$ -matrix is also diagonal  $n \times n$  matrix

$$\det_q R(q, p, \nu) \simeq \text{diag}(1, q^2(p/q)^n, \dots, (q^2(p/q)^n)^{n-1}). \quad (2.15)$$

Particular relations among parameters  $q$  and  $p$  result in the possibility of further restrictions on the  $\mathcal{A}(R_{CG})$  generators (see the next section).

The dual to  $\mathcal{A}(R_{CG})$  quasitriangular Hopf algebra with  $p = q^{1/3}$  was described in [16]. The knowledge of the twist element  $\mathcal{F}$  permits to write the entries of the  $L$ -matrices in terms of the standard  $sl_q(3)$  algebra generators. These quantities will not be used below and we end the discussion of the general properties of the quantum group and quantum algebra related to the generalized Cremmer-Gervais  $R$ -matrix  $R(q, p, \nu)$ .

### 3 Quantum linear space of $\mathcal{A}(R_{CG})$ as covariant deformed oscillator

The quantum linear space of  $\mathcal{A}(R_{CG})$  as covariant algebra (  $\mathcal{A}(R_{CG})$ -comodule algebra ), according to the general approach [8, 12], is generated by three elements  $x_i$ ,  $i = 1, 2, 3$ ;  $x_1 = A$ ,  $x_2 = K$ ,  $x_3 = A^\dagger$  which satisfy relations ( $X^t = (A, K, A^\dagger)$ )

$$R(q, p, \nu) X_1 X_2 = q X_2 X_1 = q \mathcal{P} X_1 X_2, \quad (3.1)$$

where  $X_1 X_2 = X \otimes X$  and  $X_2 X_1 = \mathcal{P} X_1 X_2$ .

In terms of the generators the defining relations are

$$KA = pqAK, \quad KA^\dagger = \frac{1}{pq} A^\dagger K, \quad AA^\dagger - p^{-2} A^\dagger A = \nu K^2. \quad (3.2)$$

The  $*$ -operation is:  $K^* = K$ ,  $(A^\dagger)^* = A$ . It is consistent with the  $*$ -operation of the quantum group  $\mathcal{A}(R_{CG})$ -coaction:

$$X^* = C X, \quad T^* = C T C \quad (T X)^* = C (T X). \quad (3.3)$$

Although there are three parameters  $q, p, \nu$  in the defining relations (3.2), this  $\mathcal{A}(R_{CG})$ -comodule algebra is nothing but a  $q$ -oscillator algebra  $\mathcal{A}_q$ . Using for the  $q$ -oscillator algebra the Arik-Coon generators (for a convenience of further identification we use  $q^2$  as the deformation parameter of  $\mathcal{A}_q$ )

$$[N, a] = -a, \quad [N, a^\dagger] = a^\dagger, \quad aa^\dagger - q^2 a^\dagger a = 1 \quad (3.4)$$

and doing a one-parameter transformation [17]

$$a(\lambda) = q^{-\lambda N} a, \quad a^\dagger(\lambda) = a^\dagger q^{-\lambda N} \quad (3.5)$$

one gets

$$a(\lambda) a^\dagger(\lambda) - q^{2(1-\lambda)} a^\dagger(\lambda) a(\lambda) = q^{-2\lambda N}. \quad (3.6)$$

Hence, the relations among generators and parameters of (3.2) and (3.5) are as follows:

$$A = a(\lambda), \quad A^\dagger = a^\dagger(\lambda), \quad K^2 = \nu^{-1} q^{-2\lambda N}, \quad p = q^{\lambda-1}. \quad (3.7)$$

The parameter of the harmonic oscillator deformation is  $q$ , while  $p = q^{\lambda-1}$  refers to a choice of the  $q$ -oscillator algebra generators. Let us point out some special values of these parameters which are of particular interest:

- 1)  $p = 1/q$ ,  $K$  is a central element, the Arik-Coon generators;
- 2)  $p = q^{-1/2}$ ,  $\lambda = 1/2$ , the Macfarlane-Biedenharn generators ;
- 3)  $p = q^{1/3}$ ,  $\lambda = 4/3$ , one could call this case the Cremmer-Gervais generators, for the quantum determinant of the preceding Sec. is central for  $\mathcal{A}(R_{CG})$ ;
- 4)  $q = 1$  (but  $p$  is arbitrary, due to appropriate limit of  $\lambda$ ) leads to the standard harmonic oscillator with nonstandard generators.

By construction, these generators (3.2) of  $\mathcal{A}_q$  are covariant with respect to the linear transformation (coaction) of the quantum group  $\mathcal{A}(R_{CG}) : X' = TX$

$$\begin{aligned} A' &= T_{11}A + T_{12}K + T_{13}A^\dagger, \\ K' &= T_{21}A + T_{22}K + T_{23}A^\dagger, \\ (A^\dagger)' &= T_{31}A + T_{32}K + T_{33}A^\dagger. \end{aligned} \quad (3.8)$$

This is a  $q$ -analogue of the Bogoliubov transformation of the canonical commutation relations. However, due to the noncommutativity of the corresponding coefficients  $T_{ij}$  the physical meaning of this transformation seems different from the standard case, where it was used to transform a general quadratic Hamiltonian in question to a simpler, canonical form. The transformation (3.9) is different also from the transformation of the  $q$ -oscillator generators with coefficients depending on the operator  $N$  [24] for in our case  $[T_{ij}, X_k] = 0$ .

To illustrate the difference in the interpretation, let us start with the simplest case 4) of the standard harmonic oscillator with nonstandard generators. This case, having trivial deformation parameter  $q = 1$ , is reduced to a pure twist  $\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1}$ . However, the generators of the quantum group are non-commuting and one can construct easily irreducible representations of  $\mathcal{A}(R_{CG})$ . To have deformation (twist) of the group of the standard Bogoliubov transformations, one can put now two out of nine generators of  $\mathcal{A}(R_{CG})$  to zero  $(T)_{2j} = 0, j = 1, 3$ . Then the quadratic relations of the remaining seven generators can be realized by operators of a Weyl pair  $UV = pVU$  with a unitary operator  $U$  and a Hermitian

$V$ . Any irreducible representation of the reduced  $\mathcal{A}(R_{CG})$  is given by these operators and four parameters of the standard Bogoliubov transformation.

Any Hamiltonian  $H(A^\dagger, A)$  (e.g.  $H_0 = A^\dagger A$ ) will be mapped under  $\mathcal{A}(R_{CG})$ -coaction into an operator in the tensor product space of the harmonic oscillator space and the space of the Weyl pair operators  $U, V$ . The irreducible representations of the complete quantum group  $\mathcal{A}(R_{CG})$  is difficult to construct, because it is noncompact  $*$ -algebra (for the compact quantum  $SU(n)$  group see [25]).

We are using this example to illustrate once more the difficulties with physical interpretation of the group-coaction deformations [14, 15], when one tries to connect this coaction with symmetry properties of the covariant system. This point is especially important while deforming the kinematical groups [26]).

## 4 Integrable models of the QISM

According to the QISM [1],[2] one can construct a variety of integrable models, related to a given solution of the YBE (an  $R$ -matrix). The importance of the corresponding spin-chain models is connected with possibilities to get different field-theoretical integrable models using different limiting procedures [1]. The extra parameters adding to the  $R$ -matrix by a twist  $F$  are quite useful in this way. It was pointed out [20], that the twist of an  $R$ -matrix

$$R(u) \rightarrow F_{21}R_{12}(u)F^{-1} \equiv R^{(F)}(u),$$

preserves the regularity property [2]: the existence of the spectral parameter value  $u = u_0$ , where  $R(u_0)$  is proportional to the permutation operator  $\mathcal{P}$ . It is obvious, that from  $R(u_0) \simeq \mathcal{P}$  it follows

$$R^{(F)}(u_0) = F_{21}R_{12}(u_0)F^{-1} \simeq \mathcal{P}.$$

This regularity property is important to get local integrals of motion (see e.g. [1,2,3]).

The Hecke condition leads to the spectral parameter dependent solution of the YBE through the Yang-Baxterization procedure [21]

$$\check{R}(u) = u\check{R} - \frac{1}{u}\check{R}^{-1} = (u - u^{-1})\check{R}_{CG} + \omega u^{-1}I. \quad (4.1)$$

Let us remind that  $\check{R}(u)$  satisfies the YBE in the braid group form.

The  $L$ -operator of the integrable spin chain coincides with the  $R$ -matrix  $R(u)$  and the density of the Hamiltonian is ( $u_0 = \pm 1$ )

$$H = \sum_n h_{n,n+1},$$

$$\begin{aligned}
h_{1,2} &\simeq \mathcal{P} \frac{d}{du} R(u)|_{u=u_0} = 2\check{R}_{CG} + \text{const.} \\
&\simeq \begin{pmatrix} q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q\nu & 0 & q/p^2 & 0 & 0 \\ 0 & p & 0 & \omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/p & 0 \\ 0 & 0 & p^2/q & 0 & -p^2\nu/q & 0 & \omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q \end{pmatrix} \\
&= \sum_{i=1}^3 q e_{ii} \otimes e_{ii} + (p e_{21} \otimes e_{12} + p^{-1} e_{12} \otimes e_{21}) + q\nu \left( p e_{12} \otimes e_{32} - (p/q)^2 e_{32} \otimes e_{12} \right) \\
&\quad + (p^2/q e_{31} \otimes e_{13} + q/p^2 e_{13} \otimes e_{31}) + \omega \sum_{a < b} e_{bb} \otimes e_{aa} .
\end{aligned}$$

Due to the fact that the twist transformation leads to the explicit expression for the new  $R$ -matrix in terms of the old one and the twist element  $\mathcal{F}$ , one can get connection between quantum scattering data (the transition matrix) of initial and deformed integrable models

$$T_t(u) = (\rho \otimes \pi)(id \otimes \Delta_t^{(N)}) \mathcal{F}_{21} \mathcal{R} \mathcal{F}^{-1}, \quad (4.2)$$

where  $\rho$  and  $\pi$  are representations of the quantum algebra in the auxiliary and quantum space respectively [20]. However, the general explicit expression for  $T_t(u)$  looks rather cumbersome. The formulae (4.2) includes  $(N-1)$  iterations of the twisted coproduct  $\Delta_t = \mathcal{F} \Delta \mathcal{F}^{-1}$ . Introducing notations

$$(id \otimes \Delta) \mathcal{F} = \mathcal{F}_{1,23}, \quad (id \otimes \Delta^{(3)}) \mathcal{F} = (id \otimes id \otimes \Delta)(id \otimes \Delta) \mathcal{F} = \mathcal{F}_{1,234} \quad \text{etc.}, \quad (4.3)$$

where the index 1 refers to the auxiliary quantum algebra (auxiliary space with irrep  $\rho$ ). Using these notations one gets for the iteration of the twisted coproduct  $\Delta_t$

$$\Delta_t^{(3)} = (id \otimes \Delta_t) \Delta_t = \mathcal{F}_{23}(id \otimes \Delta)(\mathcal{F} \Delta \mathcal{F}^{-1}) \mathcal{F}_{23}^{-1} = \mathcal{F}_{23} \mathcal{F}_{1,23} \Delta^{(3)} \mathcal{F}_{1,23}^{-1} \mathcal{F}_{23}^{-1} = X_{123} \Delta^{(3)} (X_{123})^{-1}. \quad (4.4)$$

Hence, the formulae for the transition matrix  $T_t(u)$  of the twisted (deformed) spin chain on  $(N-1)$  sites is the following

$$T_t(u) = (\rho \otimes \pi^{\otimes(N-1)}) X_{12\dots N} ((id \otimes \Delta^{(N)}) \mathcal{F}_{21} \mathcal{R} \mathcal{F}^{-1}) X_{12\dots N}^{-1}. \quad (4.5)$$

Although the iterated coproduct is a similarity transformation of the original one (4.4), the transition matrices are related by a more complicated transformation, since the factors in (4.5) are not the inverse of each other. The Yang-Baxter algebra of the quantum scattering data  $(T_t(u))_{ij}$  is more complicated than the  $SL(3)$ -spin chain [27] due to the extra non-zero

elements of  $R_{CG}(u)$  proportional to  $\nu$ . However, as in the case of the deformed (twisted)  $XXX_\xi$ -spin chain [19, 20], the spectrum of the transfer matrix  $t(u) = \text{tr} T_t(u)$  is expected to be the same as for the  $SL(3)$ -spin chain, taking into account the changed eigenvalues of the diagonal elements of  $T_t(u)$  on the reference (vacuum) state  $(0, 0, 1)^t$ . The Bethe equations defining the two sets of quasimomenta have similar structure with obvious changes due to the parameter  $p$  [27]. The second parameter  $\nu$  enters into the eigenvectors and adjoint vectors.

## 5 Conclusion

The twist transformation in the theory of quantum groups preserves the algebraic sector of the twisted Hopf algebra. Hence the representations of the twisted algebra are the same. However, the change of the coproduct has deep consequences for the R-matrix structure and for the corresponding integrable models, which rely on the tensor product of these representations. The generalized Cremmer-Gervais  $R$ -matrix, being a twist of the standard one, leads to more complicated integrable systems. Two of these systems were studied in this paper: i) the covariant  $q$ -oscillator algebra, and ii) an integrable  $sl(3)$  spin chain. The covariance of the  $q$ -oscillator algebra under the  $q$ -Bogoliubov transformation results in preservation of the  $\mathcal{A}_q$  structure in an *extended* dynamical system. We hope that further research and explicit knowledge of the twist element permits to calculate most of the characteristics of the twisted models in terms of the initial ones.

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